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Non-linear wave modulation near the marginal state of instability

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Abstract. The slow-amplitude modulation of a weakly non-linear purely dispersive quasimonochromatic wave solution to a certain system of quasilinear partial differential equations is investigated. It is shown that, near the marginal state of modulational instability, and provided the system satisfies certain conditions, the complex amplitude of the wave is governed by a modified form of the non-linear Schrödinger equation that involves higher-order non-linearities.

1. Introduction

Many purely dispersive physical systems are described by the class of quasilinear partial differential equations

$$A(U)\partial U/\partial t + B(U)\partial U/\partial x + C(U) = 0 \tag{1.1}$$

where t and x are time and space coordinates, respectively, $U=(u_i)$ is an n-component column vector function of t and x, and the $n\times n$ matrices $A=(a_{ij})$, $B=(b_{ij})$ and the n-component column vector $C=(c_i)$ are functions of the u_i, all these quantities being real. Inoue and Matsumoto (1974) have shown that under certain circumstances the slow-amplitude modulation of a weakly non-linear quasimonochromatic (carrier) wave solution to (1.1) is governed by the non-linear Schrödinger (NS) equation

$$\mathrm{i}\frac{\partial\varphi}{\partial\tau_2} + p\frac{\partial^2\varphi}{\partial\xi_1^2} = q|\varphi|^2\varphi. \tag{1.2}$$

In (1.2), $\tau_2=\varepsilon^2 t$, $\xi_1=\varepsilon(x-V_g t)$, φ, V_g and κ are the complex amplitude, group velocity and wavenumber of the carrier wave, respectively, $p=\frac{1}{2}\,\mathrm{d}V_g/\mathrm{d}\kappa$ and q are real functions of κ, and ε is a small parameter that characterises the modulation and the non-linearity.

The criterion for the modulational instability of the carrier is $pq<0$ (Taniuti and Yajima 1969). For many physical systems pq has just one real zero at some critical wavenumber κ_c and so the carrier is marginally modulationally unstable at $\kappa=\kappa_c$. Inoue and Matsumoto (1974) implicitly assumed that p and q were O(1) quantities and so their derivation of (1.2) is not valid when κ is near κ_c. The purpose of this paper is to find the governing equation for φ that replaces (1.2) near the marginal state.

Hasimoto and Ono (1972) considered the modulation of Stokes waves (i.e. gravity waves on water of uniform depth) away from the marginal state and obtained (1.2) as the governing equation for φ. Kakutani and Michihiro (1983) reconsidered this problem and argued that near the marginal state a different ordering should be used to intensify the effect of the non-linearity. We have discussed this argument in a previous paper (Parkes 1987a). By assuming that the effect of the non-linearity is of $O(\varepsilon^{1/2})$ instead of $O(\varepsilon)$, Kakutani and Michihiro derived a governing equation for φ near the marginal state of the form

$$i\frac{\partial\varphi}{\partial\tau_2}+p\frac{\partial^2\varphi}{\partial\xi_1^2}=q_1|\varphi|^2\varphi+q_2|\varphi|^4\varphi+iq_3\varphi\frac{\partial}{\partial\xi_1}|\varphi|^2+iq_4|\varphi|^2\frac{\partial\varphi}{\partial\xi_1} \tag{1.3}$$

where $q_1(=q/\varepsilon)$, q_2, q_3 and q_4 are $O(1)$ real functions of κ, which is assumed to be such that $\kappa-\kappa_c$ is of $O(\varepsilon)$. We shall refer to (1.3) as the modified non-linear Schrödinger (MNS) equation. This was also derived formally by Parkes (1987a) for modulations near the marginal state for an arbitrary system in which a scalar-dependent variable u satisfies an equation of the form $\mathscr{L}u=N$, where $\mathscr{L}$ is a linear operator involving the differential operators $\partial/\partial t$ and $\partial/\partial x$, and N represents all the non-linear terms. In Parkes (1987b) we considered a particular example of (1.1), namely a 4×4 system describing ion acoustic wave propagation in a plasma, and, using Kakutani and Michihiro's ordering, we obtained (1.3) again for modulations near the marginal state.

In this paper we apply the methodology developed in Parkes (1987b) to the general system (1.1). We show that, under certain restrictions, the MNS equation is the governing equation for φ near the marginal state. In § 2 we show how to apply the derivative expansion perturbation procedure to (1.1) and describe the method for solving the resulting hierarchy of equations. In § 3 we give the details of the solution and derive the conditions that A, B and C have to satisfy in order that φ is governed by the MNS equation. In § 4 we compare our results with those of Inoue and Matsumoto (1974) for modulations away from the marginal state.

2. Method of solution

The unperturbed state of the system described by (1.1) corresponds to the constant solution $U=U^{(0)}$ for which $C(U^{(0)})=0$. Using this it is convenient to rewrite (1.1) in the form

$$LU=M \tag{2.1}$$

where

$$L=A\partial/\partial t+B\partial/\partial x+\nabla C_0 \tag{2.2}$$

$$M=-C+(\nabla C_0)U \tag{2.3}$$

and ∇C_0 is defined by $(\nabla C_0)_{ij}=\partial c_i/\partial u_j$ evaluated at $U=U^{(0)}$.

The derivative expansion procedure (Kawahara 1973) is applied to the system (2.1) by introducing the extended set of independent variables

$$t_0=t \qquad x_0=x \qquad \tau_i=\varepsilon^i t \qquad \xi_i=\varepsilon^i(x-V_g t) \qquad i=1,2,\ldots,N$$

where ε is a small parameter characterising the slow modulation. As in Parkes (1987a, b) it is sufficient to take $N=2$ here. Thus defined t_0, x_0 are the variables appropriate to the 'fast' oscillations of the carrier, and τ_1, ξ_1, τ_2, ξ_2 are 'slow' variables appropriate to the slow modulations in a reference frame moving with the group velocity V_g. (An explicit expression for V_g is given later.) The time and space derivatives in (2.2) may now be expressed as the derivative expansions

$$\frac{\partial}{\partial t} \equiv -\omega \frac{\partial}{\partial \theta} + \varepsilon \left(\frac{\partial}{\partial \tau_1} - V_g \frac{\partial}{\partial \xi_1} \right) + \varepsilon^2 \left(\frac{\partial}{\partial \tau_2} - V_g \frac{\partial}{\partial \xi_2} \right)$$

$$\frac{\partial}{\partial x} \equiv \kappa \frac{\partial}{\partial \theta} + \varepsilon \frac{\partial}{\partial \xi_1} + \varepsilon^2 \frac{\partial}{\partial \xi_2} \tag{2.4}$$

where $\theta(= \kappa x_0 - \omega t_0)$, ω and κ are, to lowest order, the phase, angular frequency and wavenumber of the fast oscillations.

Inoue and Matsumoto (1974) considered the non-marginal state by assuming that the non-linearity is of $O(\varepsilon)$. In order to investigate the behaviour of the slow modulations near marginal instability we intensify the non-linear effects by assuming the non-linearity to be of $O(\varepsilon^{1/2})$, as in Kakutani and Michihiro (1983). As in Parkes (1987b) we write U as

$$U = U^{(0)} + \sum_{i=1}^{6} \varepsilon^{i/2} U^{(i)}(\theta, \tau_1, \xi_1, \tau_2, \xi_2) + O(\varepsilon^{7/2}). \tag{2.5}$$

Substitution of (2.4), (2.5) and the Taylor series expansions for A, B and C about $U = U^{(0)}$ (given in appendix 1) into (2.2) and (2.3) gives

$$L = \sum_{i=0}^{5} \varepsilon^{i/2} L_i + O(\varepsilon^3) \tag{2.6}$$

$$M = \sum_{i=2}^{6} \varepsilon^{i/2} M_i(\theta, \tau_1, \xi_1, \tau_2, \xi_2) + O(\varepsilon^{7/2}) \tag{2.7}$$

where

$$L_i = (-\omega A_i + \kappa B_i) \frac{\partial}{\partial \theta} + \sum_{j=1}^{k} \left(A_{i-2j} \frac{\partial}{\partial \tau_j} + (B_{i-2j} - V_g A_{i-2j}) \frac{\partial}{\partial \xi_j} \right)$$

and

$$k = \begin{cases} \frac{1}{2} i & i \text{ even} \\ \frac{1}{2}(i-1) & i \text{ odd.} \end{cases}$$

The A_i, B_i and M_i are given in appendix 1. Substituting (2.5)-(2.7) into (2.1) and equating like powers of ε, we obtain the hierarchy of equations

$$O(\varepsilon^{l/2}):\ L_0 U^{(l)} = \begin{cases} 0 & l = 1 \\ M_l - \sum_{k=1}^{l-1} L_k U^{(l-k)} & l = 2, \ldots, 6. \end{cases} \tag{2.8}$$

For $l = 1$ we assume that the solution to (2.8) is the quasimonochromatic wave

$$U^{(1)} = \varphi(\tau_1, \xi_1, \tau_2, \xi_2) K \exp(\mathrm{i}\theta) + \mathrm{cc} \tag{2.9}$$

where φ is a complex scalar function and K is a constant column vector. Here, and subsequently, cc is used to denote the complex conjugate of all the preceding terms. Substituting (2.9) into (2.8) we deduce that there is a non-trivial solution for K provided ω and κ satisfy

$$\mathscr{D}(\omega, \kappa) \equiv \det\{D_0(\omega, \kappa)\} = 0 \tag{2.10}$$

where

$$D_0(\omega, \kappa) = -\mathrm{i}\omega A_0 + \mathrm{i}\kappa B_0 + \nabla C_0 \equiv (d_{ij}).$$

Usually it is found that rank $\{D_0(\omega, \kappa)\} = n-1$. Equation (2.10) is the linear dispersion relation which, for purely dispersive waves, is satisfied by real values of $\omega = \omega(\kappa)$ and κ. We shall assume that $\mathscr{D}(j\omega, j\kappa) \neq 0$ for $j = 2, 3, \ldots$, so that the inverse of the matrix $D_0(j\omega, j\kappa)$ exists for these values of j.

If D_{ij} is the cofactor of d_{ij} then for a given r†

$$d_{ij}D_{rj} = 0 \qquad i = 1, \ldots, n \tag{2.11}$$

since for $i = r$ the left-hand side is just $\det\{D_0(\omega, \kappa)\}$ which is zero by (2.10), while for other values of i the left-hand side is an alien cofactor expansion and so is identically zero. Similarly, for a given s

$$d_{ij}D_{is} = 0 \qquad j = 1, \ldots, n. \tag{2.12}$$

As K satisfies $d_{ij}K_j = 0$, $i = 1, \ldots, n$, we may choose $K_i = D_{ri}$, where r is selected so that K is not the zero vector. In appendix 2 equations (2.11) and (2.12) are used in the derivation of an expression for the group velocity $V_g \equiv \mathrm{d}\omega/\mathrm{d}\kappa$, namely (A2.3). For a purely dispersive system this expression is real.

Explicit solutions to (2.8) for $l > 1$ are given in § 3. Here we summarise the method of solution and obtain non-secular conditions. We find that, for each $l > 1$, (2.8) may be written

$$L_0 U^{(l)} = \tilde{U}_0^{(l)} + \left(\sum_{k=1}^{l} \tilde{U}_k^{(l)} \exp(\mathrm{i}k\theta) + \mathrm{cc} \right) \tag{2.13}$$

where the $\tilde{U}_k^{(l)}$, $k = 0, \ldots, l$, are independent of θ and are determined by the solutions $U^{(i)}$, $i = 1, \ldots, l-1$, to previous equations in the hierarchy. As we require solutions involving no secular terms we assume a solution to (2.13) of the form

$$U^{(l)} = U_0^{(l)} + \left(\sum_{k=1}^{l} U_k^{(l)} \exp(\mathrm{i}k\theta) + \mathrm{cc} \right)$$

where the $U_k^{(l)}$, $k = 0, \ldots, l$, are independent of θ. This assumption imposes up to two conditions at each order, namely (2.15) and (2.18) below, which may be regarded as 'non-secular conditions'. The $U_k^{(l)}$ are determined as follows.

The vector $U_0^{(l)}$ satisfies $L_0 U_0^{(l)} = \tilde{U}_0^{(l)}$, from which we obtain

$$(\nabla C_0) U_0^{(l)} = \tilde{U}_0^{(l)}. \tag{2.14}$$

Suppose rank $\{\nabla C_0\} = m$. If $m = n$, ∇C_0 is non-singular and the solution to (2.14) is simply $U_0^{(l)} = (\nabla C_0)^{-1} \tilde{U}_0^{(l)}$. However, for $m < n$, ∇C_0 is singular and (2.14) has a solution only if

$$\mathrm{rank}\{(\nabla C_0, \tilde{U}_0^{(l)})\} = \mathrm{rank}\{\nabla C_0\} \tag{2.15}$$

where $(\nabla C_0, \tilde{U}_0^{(l)})$ is the $n \times (n+1)$ augmented matrix whose $(n+1)$th column consists of the components of $\tilde{U}_0^{(l)}$. The condition (2.15) can be stated more conveniently as follows. We may re-order the equations in the system (2.1) and re-order the components of U so that $(\nabla C_0)_{ij} = 0$, $i = 1, \ldots, n-m$; $j = 1, \ldots, n$, and that the $m \times m$ matrix

† Here and elsewhere, except when indicated otherwise, the summation convention is used with the repeated subscript running from 1 to n. The summation convention will not apply where the repeated subscript is r or s.

$(\nabla C_0)_{ij}$, $i = n-m+1, \ldots, n$; $j = n-m+1, \ldots, n$, is non-singular. Then (2.15) implies that

$$\tilde{u}_{0i}^{(l)} = 0 \qquad i = 1, \ldots, n-m \tag{2.16}$$

where $\tilde{u}_{0i}^{(l)}$ is the ith component of $\tilde{U}_0^{(l)}$. The solution to (2.14) may now be written

$$u_{0i}^{(l)} = \begin{cases} \lambda_i^{(l)} & i = 1, \ldots, n-m \\ \sum_{j=n-m+1}^{n} (\nabla C_0)_{ij}^{-1} \tilde{u}_{0j}^{(l)} + \sum_{j=1}^{n-m} e_{ij} \lambda_j^{(l)} & i = n-m+1, \ldots, n \end{cases}$$

where

$$e_{ij} = -\sum_{k=n-m+1}^{n} (\nabla C_0)_{ik}^{-1} (\nabla C_0)_{kj} \qquad i = n-m+1, \ldots, n;\, j = 1, \ldots, n-m$$

and the $\lambda_i^{(l)}$, $i = 1, \ldots, n-m$, are arbitrary real functions of the slow variables.

The vector $U_1^{(l)}$ satisfies $L_0 U_1^{(l)} \exp(\mathrm{i}\theta) = \tilde{U}_1^{(l)} \exp(\mathrm{i}\theta)$, from which we obtain

$$D_0(\omega, \kappa) U_1^{(l)} = \tilde{U}_1^{(l)}. \tag{2.17}$$

Equation (2.17) has a solution only if

$$\operatorname{rank}\{(D_0(\omega, \kappa), \tilde{U}_1^{(l)}\} = \operatorname{rank}\{D_0(\omega, \kappa)\}. \tag{2.18}$$

This condition can be stated more conveniently as follows. Writing (2.17) as $d_{ij} u_{1j}^{(l)} = \tilde{u}_{1i}^{(l)}$, and choosing s so that the vector with components D_{is}, $i = 1, \ldots, n$, is not the zero vector, we have $D_{is} d_{ij} u_{1j}^{(l)} = D_{is} \tilde{u}_{1i}^{(l)}$. Then, on using (2.12), we obtain a condition equivalent to (2.18), namely

$$D_{is} \tilde{u}_{1i}^{(l)} = 0. \tag{2.19}$$

The solution to (2.17) consists of a particular solution plus a solution to the homogeneous version of (2.17). We shall ignore the latter as it may be absorbed into (2.9) by a suitable redefinition of φ.

The vectors $U_k^{(l)}$, $k = 2, \ldots, l$, satisfy $L_0 U_k^{(l)} \exp(\mathrm{i}k\theta) = \tilde{U}_k^{(l)} \exp(\mathrm{i}k\theta)$, from which we obtain $D_0(k\omega, k\kappa) U_k^{(l)} = \tilde{U}_k^{(l)}$. The solution is simply $U_k^{(l)} = [D_0(k\omega, k\kappa)]^{-1} \tilde{U}_k^{(l)}$.

3. Derivation of the modified non-linear Schrödinger equation

Following Inoue and Matsumoto (1974) we classify the system (1.1) as type 1 if $m = n$ (or, equivalently, $\det\{\nabla C_0\} \neq 0$) or type 2 if $m < n$ (or, equivaletly, $\det\{\nabla C_0\} = 0$), where $m = \operatorname{rank}\{\nabla C_0\}$. The derivation of the MNS equation is more complicated for a type-2 system. In this section we present the calculation for this case and then comment briefly on the calculation for a type-1 system.

At $O(\varepsilon)$ we have $\tilde{U}_0^{(2)} = \tilde{\alpha}^{(2)} |\varphi|^2$, $\tilde{U}_1^{(2)} = 0$ and $\tilde{U}_2^{(2)} = \tilde{\beta}^{(2)} \varphi^2$, where the vectors $\tilde{\alpha}^{(2)}$ and $\tilde{\beta}^{(2)}$ are given in appendix 3. The condition (2.16) applied to $\tilde{U}_0^{(2)}$ implies that

$$\tilde{\alpha}_i^{(2)} = 0 \qquad i = 1, \ldots, n-m. \tag{3.1}$$

The solution to (2.8) is $U_0^{(2)} = \alpha^{(2)} |\varphi|^2$, $U_1^{(2)} = 0$ and $U_2^{(2)} = \beta^{(2)} \varphi^2$, where

$$\alpha_i^{(2)} = \begin{cases} \lambda_i^{(2)} & i = 1, \ldots, n-m \\ \sum_{j=n-m+1}^{n} (\nabla C_0)_{ij}^{-1} \tilde{\alpha}_j^{(2)} + \sum_{j=1}^{n-m} e_{ij} \lambda_j^{(2)} & i = n-m+1, \ldots, n \end{cases} \tag{3.2}$$

$\beta^{(2)} = [D_0(2\omega, 2\kappa)]^{-1} \tilde{\beta}^{(2)}$, and the $\lambda_i^{(2)}$, $i = 1, \ldots, n-m$, are arbitrary real functions of the slow variables which will be determined at the $O(\varepsilon^2)$ level of the hierarchy.

At $O(\varepsilon^{3/2})$ we have $\tilde{U}_0^{(3)}=0$, $\tilde{U}_2^{(3)}=0$, $\tilde{U}_3^{(3)}=\tilde{\delta}^{(3)}\varphi^3$ and

$$\tilde{U}_1^{(3)}=\tilde{\alpha}^{(3)}\partial\varphi/\partial\tau_1+\tilde{\beta}^{(3)}\partial\varphi/\partial\xi_1+\tilde{\gamma}^{(3)}|\varphi|^2\varphi \tag{3.3}$$

where the vectors $\tilde{\gamma}^{(3)}$ and $\tilde{\delta}^{(3)}$ are given in appendix 3, and

$$\tilde{\alpha}_i^{(3)}=-(A_0)_{ij}D_{rj} \tag{3.4}$$

$$\tilde{\beta}_i^{(3)}=(V_gA_0-B_0)_{ij}D_{rj}. \tag{3.5}$$

The condition (2.19) applied to $\tilde{U}_1^{(3)}$ implies that

$$\mathrm{i}\partial\varphi/\partial\tau_1=q|\varphi|^2\varphi \tag{3.6}$$

where

$$q=\mathrm{i}D_{is}\tilde{\gamma}_i^{(3)}/(\nu D_{rs})$$

and we have used (A2.2) and (A2.3) from appendix 2. Note that q involves the as yet undetermined quantities $\lambda_i^{(2)}$, $i=1,\ldots,n-m$. The solution to (2.8) is $u_{0i}^{(3)}=\lambda_i^{(3)}$, $U_2^{(3)}=0$, $U_3^{(3)}=\delta^{(3)}\varphi^3$ and

$$U_1^{(3)}=\alpha^{(3)}\partial\varphi/\partial\tau_1+\beta^{(3)}\partial\varphi/\partial\xi_1+\gamma^{(3)}|\varphi|^2\varphi \tag{3.7}$$

where $\delta^{(3)}=[D_0(3\omega,3\kappa)]^{-1}\tilde{\delta}^{(3)}$, the $\lambda_i^{(3)}$, $i=1,\ldots,n-m$, are arbitrary real functions of the slow variables which will be determined at the $O(\varepsilon^{5/2})$ level of the hierarchy, and

$$\lambda_i^{(3)}=\sum_{j=1}^{n-m}e_{ij}\lambda_j^{(3)} \qquad i=n-m+1,\ldots,n.$$

Using (2.11) and (3.5) we may write $\tilde{\beta}_i^{(3)}=-\mathrm{i}d_{ij}\,\mathrm{d}(D_{rj})/\mathrm{d}\kappa$. As $\beta^{(3)}$ satisfies $d_{ij}\beta_j^{(3)}=\tilde{\beta}_i^{(3)}$, a possible solution for $\beta^{(3)}$ is

$$\beta_i^{(3)}=-\mathrm{i}\,\mathrm{d}(D_{ri})/\mathrm{d}\kappa. \tag{3.8}$$

After considering the implications of (3.6) we shall find that $\gamma^{(3)}$ satisfies (3.15) and that $\alpha^{(3)}$ is not required at this order. It reappears at $O(\varepsilon^{5/2})$ and we find that it satisfies (3.25).

At $O(\varepsilon^2)$ we have

$$\tilde{U}_0^{(4)}=(\tilde{\alpha}^{(4)}\varphi\partial\varphi^*/\partial\tau_1+\tilde{\beta}^{(4)}\varphi\partial\varphi^*/\partial\xi_1+\mathrm{cc})+\tilde{\gamma}^{(4)}|\varphi|^4$$

where * denotes the complex conjugate,

$$\tilde{\beta}_i^{(4)}=(V_gA_0-B_0)_{ij}\alpha_j^{(2)}+f_i$$

and f_i and the vectors $\tilde{\alpha}^{(4)}$ and $\tilde{\gamma}^{(4)}$ are given in appendix 3. Let us assume *a priori* that $\partial\varphi/\partial\tau_1=0$; then the condition (2.16) applied to $\tilde{U}_0^{(4)}$ is satisfied if

$$\tilde{\beta}_i^{(4)}=0 \qquad i=1,\ldots,n-m \tag{3.9}$$

$$\tilde{\gamma}_i^{(4)}=0 \qquad i=1,\ldots,n-m. \tag{3.10}$$

On incorporating the expression (3.2) for $\alpha^{(2)}$, we may rearrange (3.9) to give

$$\lambda_i^{(2)}=\sum_{j=1}^{n-m}(F)_{ij}^{-1}\left(f_j-\sum_{k=n-m+1}^{n}g_{jk}\tilde{\alpha}_k^{(2)}\right) \qquad i=1,\ldots,n-m \tag{3.11}$$

where

$$(F)_{ij}=(-V_gA_0+B_0)_{ij}+\sum_{k=n-m+1}^{n}(-V_gA_0+B_0)_{ik}e_{kj} \qquad i=1,\ldots,n-m;\, j=1,\ldots,n-m$$

and

$$g_{ij}=\sum_{k=n-m+1}^{n}(-V_gA_0+B_0)_{ik}(\nabla C_0)^{-1}_{kj} \qquad i=1,\ldots,n-m;\, j=n-m+1,\ldots,n.$$

It is easily shown that

$$\tilde{\beta}_i^{(4)}-\tilde{\beta}_i^{(4)*}=f_i-f_i^*=\mathrm{i}\,\mathrm{d}\tilde{\alpha}_i^{(2)}/\mathrm{d}\kappa \tag{3.12}$$

and so, in view of (3.1), it follows that f_i is real for $i=1,\ldots,n-m$, and hence that the right-hand side of (3.11) is real, as required.

Let us now return to the $\mathrm{O}(\varepsilon^{3/2})$ problem. In (3.6) q is now completely determined. It is precisely the q that appears in the NS equation (1.2) derived on the assumption that κ is not near κ_c and that q is of O(1)—see equation (2.35) of Inoue and Matsumoto (1974). Here, however, we are considering the marginal state and we assume that $\Delta\kappa\equiv\kappa-\kappa_c$ is of $\mathrm{O}(\varepsilon)$ and write $q=\varepsilon q_1$, where q_1 is of O(1) and is given approximately by

$$q_1=\frac{\Delta\kappa}{\varepsilon}\left(\frac{\mathrm{d}q}{\mathrm{d}\kappa}\right)_{\kappa=\kappa_c}. \tag{3.13}$$

Hence at $\mathrm{O}(\varepsilon^{3/2})$ equation (3.6) becomes

$$\partial\varphi/\partial\tau_1=0 \tag{3.14}$$

thus justifying our *a priori* assumption. The right-hand side of (3.6) must be shifted to the corresponding non-secular condition at $\mathrm{O}(\varepsilon^{5/2})$. This is readily achieved by including a term $\mathrm{i}q_1\tilde{\alpha}^{(3)}|\varphi|^2\varphi$ in $\tilde{U}_1^{(5)}$ (see (3.21)) and revising (3.3) to

$$\tilde{U}_1^{(3)}=\tilde{\beta}^{(3)}\partial\varphi/\partial\xi_1+(\tilde{\gamma}^{(3)}-\mathrm{i}q\tilde{\alpha}^{(3)})|\varphi|^2\varphi.$$

Then (3.7) is revised to

$$U_1^{(3)}=\beta^{(3)}\partial\varphi/\partial\xi_1+\gamma^{(3)}|\varphi|^2\varphi$$

where $\gamma^{(3)}$ is a particular solution to

$$d_{ij}\gamma_j^{(3)}=\tilde{\gamma}_i^{(3)}-\mathrm{i}q\tilde{\alpha}_i^{(3)}. \tag{3.15}$$

Continuing now with the $\mathrm{O}(\varepsilon^2)$ problem, and hereafter incorporating (3.11) and (3.14) into our results, we have $\tilde{U}_3^{(4)}=0$,

$$\tilde{u}_{1i}^{(4)}=r_{ijk}D_{rj}\lambda_k^{(3)}\varphi$$

$$\tilde{U}_2^{(4)}=\tilde{\delta}^{(4)}\varphi\partial\varphi/\partial\xi_1+\tilde{\varepsilon}^{(4)}|\varphi|^2\varphi^2$$

where r_{ijk} and the vector $\tilde{\varepsilon}^{(4)}$ are given in appendix 3, and

$$\tilde{\delta}_i^{(4)}=2(V_gA_0-B_0)_{ij}\beta_j^{(2)}-\mathrm{i}\,\mathrm{d}\tilde{\beta}_i^{(2)}/\mathrm{d}\kappa. \tag{3.16}$$

As $\tilde{U}_4^{(4)}$ plays no part in the subsequent calculations, it is not given here. The condition (2.19) applied to $\tilde{U}_1^{(4)}$, together with (A2.2), implies that

$$r_{ijk}D_{ij}\lambda_k^{(3)}=0. \tag{3.17}$$

The solution to (2.8) is

$$U_0^{(4)}=(\beta^{(4)}\varphi\partial\varphi^*/\partial\xi_1+\mathrm{cc})+\gamma^{(4)}|\varphi|^4 \qquad U_1^{(4)}=\zeta^{(4)}\varphi$$

$$U_2^{(4)}=\delta^{(4)}\varphi\partial\varphi/\partial\xi_1+\varepsilon^{(4)}|\varphi|^2\varphi^2 \qquad U_3^{(4)}=0$$

where

$$\beta_i^{(4)}=\begin{cases}\mu_i^{(4)} & i=1,\ldots,n-m\\ \displaystyle\sum_{j=n-m+1}^{n}(\nabla C_0)_{ij}^{-1}\tilde\beta_j^{(4)}+\sum_{j=1}^{n-m}e_{ij}\mu_j^{(4)} & i=n-m+1,\ldots,n\end{cases}\tag{3.18}$$

$$\gamma_i^{(4)}=\begin{cases}\lambda_i^{(4)} & i=1,\ldots,n-m\\ \displaystyle\sum_{j=n-m+1}^{n}(\nabla C_0)_{ij}^{-1}\tilde\gamma_j^{(4)}+\sum_{j=1}^{n-m}e_{ij}\lambda_j^{(4)} & i=n-m+1,\ldots,n\end{cases}\tag{3.19}$$

$\delta^{(4)}=[D_0(2\omega,2\kappa)]^{-1}\tilde\delta^{(4)}$, $\varepsilon^{(4)}=[D_0(2\omega,2\kappa)]^{-1}\tilde\varepsilon^{(4)}$ and $\zeta^{(4)}$ is a particular solution to

$$d_{ij}\zeta_j^{(4)}=r_{ijk}D_{rj}\lambda_k^{(3)}.\tag{3.20}$$

The $\mu_i^{(4)}$, $i=1,\ldots,n-m$, are arbitrary complex functions of the slow variables, and the $\lambda_i^{(4)}$, $i=1,\ldots,n-m$, are arbitrary real functions of the slow variables. They will be determined at the $O(\varepsilon^3)$ level of the hierarchy. By comparing (3.16) with the derivative with respect to κ of the relation $D_0(2\omega,2\kappa)\beta^{(2)}=\tilde\beta^{(2)}$, we find that $\tilde\delta^{(4)}=-\mathrm{i}D_0(2\omega,2\kappa)\,\mathrm{d}\beta^{(2)}/\mathrm{d}\kappa$ and hence that

$$\delta^{(4)}=-\mathrm{i}\,\mathrm{d}\beta^{(2)}/\mathrm{d}\kappa.$$

Also, from (3.12), it follows that $(\nabla C_0)(\beta^{(4)}-\beta^{(4)*}-\mathrm{i}\,\mathrm{d}\alpha^{(2)}/\mathrm{d}\kappa)=0$ and hence that

$$\beta_i^{(4)}-\beta_i^{(4)*}-\mathrm{i}\,\mathrm{d}\alpha_i^{(2)}/\mathrm{d}\kappa=\mathrm{i}\sigma_i$$

where, on using (3.2) and (3.18),

$$\mathrm{i}\sigma_i=\begin{cases}\mu_i^{(4)}-\mu_i^{(4)*}-\mathrm{i}\,\mathrm{d}\lambda_i^{(2)}/\mathrm{d}\kappa & i=1,\ldots,n-m\\ \mathrm{i}\displaystyle\sum_{j=1}^{n-m}e_{ij}\sigma_j & i=n-m+1,\ldots,n\end{cases}$$

and the σ_i, $i=1,\ldots,n-m$, are real functions of the slow variables.

At $O(\varepsilon^{5/2})$ we have

$$\tilde u_{0i}^{(5)}=[-(A_0)_{ij}(\partial/\partial\tau_1)+(V_gA_0-B_0)_{ij}(\partial/\partial\xi_1)]\lambda_j^{(3)}$$
$$+\{s_{ijk}[D_{rk}^*\zeta_j^{(4)}+D_{rj}\zeta_k^{(4)*}+\alpha_j^{(2)}\lambda_k^{(3)}]+s_{ijkl}D_{rj}D_{rk}^*\lambda_l^{(3)}+\mathrm{cc}\}|\varphi|^2$$
$$\tilde U_1^{(5)}=\tilde\alpha^{(3)}\partial\varphi/\partial\tau_2+\tilde\beta^{(3)}\partial\varphi/\partial\xi_2+\tilde\gamma^{(5)}\partial^2\varphi/\partial\xi_1^2+\tilde\delta^{(5)}\varphi\partial(|\varphi|^2)/\partial\xi_1$$
$$+\tilde\varepsilon^{(5)}|\varphi|^2\partial\varphi/\partial\xi_1+\tilde\zeta^{(5)}|\varphi|^4\varphi+\mathrm{i}q_1\tilde\alpha^{(3)}|\varphi|^2\varphi\tag{3.21}$$

where s_{ijk}, s_{ijkl}, and the vectors $\tilde\delta^{(5)}$, $\tilde\varepsilon^{(5)}$ and $\tilde\zeta^{(5)}$ are given in appendix 3, and

$$\tilde\gamma_i^{(5)}=\mathrm{i}(-V_gA_0+B_0)_{ij}\,\mathrm{d}(D_{rj})/\mathrm{d}\kappa.\tag{3.22}$$

If we take the $\lambda_i^{(3)}=0$, $i=1,\ldots,n-m$, then $\zeta^{(4)}=0$ is a solution to (3.20), and both (3.17) and the condition (2.16) applied to $\tilde U_0^{(5)}$ are satisfied. Now it follows that $U_0^{(3)}=0$ and $\tilde U_0^{(5)}=0$, although the latter result is not used subsequently. The condition (2.19) applied to $\tilde U_1^{(5)}$ gives the MNS equation (1.3) with q_1 given by (3.13) and

$$\begin{aligned}&p=[(-V_gA_0+B_0)_{ij}D_{is}\,\mathrm{d}(D_{rj})/\mathrm{d}\kappa]/(\nu D_{rs})\\ &q_2=\mathrm{i}D_{is}\tilde\zeta_i^{(5)}/(\nu D_{rs})\qquad q_3=D_{is}\tilde\delta_i^{(5)}/(\nu D_{rs})\qquad q_4=D_{is}\tilde\varepsilon_i^{(5)}/(\nu D_{rs}).\end{aligned}\tag{3.23}$$

In appendix 2 we show that p may be identified as $\frac{1}{2}\,\mathrm{d}V_g/\mathrm{d}\kappa$. For a purely dispersive system p is real.

Although we have obtained the desired MNS equation, q_2, q_3 and q_4 are not yet completely determined since $\tilde{\delta}^{(5)}$ and $\tilde{\varepsilon}^{(5)}$ involve the $\mu_i^{(4)}$ and $\mu_i^{(4)*}$, $i=1,\ldots,n-m$, and $\tilde{\zeta}^{(5)}$ involves the $\lambda_i^{(4)}$, $i=1,\ldots,n-m$. In order to determine the $\lambda_i^{(4)}$, $\mu_i^{(4)}$ and $\mu_i^{(4)*}$ we need to go to the next order, and there we find that the only information we need about the solution at $O(\varepsilon^{5/2})$ is $U_1^{(5)}$. This is given by

$$U_1^{(5)}=\alpha^{(3)}\partial\varphi/\partial\tau_2+\beta^{(3)}\partial\varphi/\partial\xi_2+\gamma^{(5)}\partial^2\varphi/\partial\xi_1^2+\delta^{(5)}\varphi\partial(|\varphi|^2)/\partial\xi_1$$
$$+\varepsilon^{(5)}|\varphi|^2\partial\varphi/\partial\xi_1+\zeta^{(5)}|\varphi|^4\varphi+\mathrm{i}q_1\alpha^{(3)}|\varphi|^2\varphi. \tag{3.24}$$

Here $\beta^{(3)}$ is given by (3.8), and on using (1.3) to eliminate the $\partial\varphi/\partial\tau_2$ terms in (3.21) and (3.24) we deduce that the other vector coefficients in (3.24) satisfy the following equations:

$$d_{ij}(\gamma_j^{(5)}+\mathrm{i}p\alpha_j^{(3)})=\tilde{\gamma}_i^{(5)}+\mathrm{i}p\tilde{\alpha}_i^{(3)} \tag{3.25a}$$
$$d_{ij}(\delta_j^{(5)}+q_3\alpha_j^{(3)})=\tilde{\delta}_i^{(5)}+q_3\tilde{\alpha}_i^{(3)} \tag{3.25b}$$
$$d_{ij}(\varepsilon_j^{(5)}+q_4\alpha_j^{(3)})=\tilde{\varepsilon}_i^{(5)}+q_4\tilde{\alpha}_i^{(3)} \tag{3.25c}$$
$$d_{ij}(\zeta_j^{(5)}-\mathrm{i}q_2\alpha_j^{(3)})=\tilde{\zeta}_i^{(5)}-\mathrm{i}q_2\tilde{\alpha}_i^{(3)}. \tag{3.25d}$$

Combining (3.4) and (3.22), and using (2.11), we find that $\tilde{\gamma}_i^{(5)}+\mathrm{i}p\tilde{\alpha}_i^{(3)}=-\frac{1}{2}d_{ij}\,\mathrm{d}^2(D_{rj})/\mathrm{d}\kappa^2$. Hence a particular solution to (3.25a) is

$$\gamma_i^{(5)}+\mathrm{i}p\alpha_i^{(3)}=-\tfrac{1}{2}\,\mathrm{d}^2(D_{ri})/\mathrm{d}\kappa^2.$$

At $O(\varepsilon^3)$ we have, on eliminating a term in $\partial\varphi/\partial\tau_2$ by using (1.3),

$$\tilde{U}_0^{(6)}=\left[\tilde{\beta}^{(4)}\varphi\frac{\partial\varphi^*}{\partial\xi_2}+\tilde{\gamma}^{(6)}\frac{\partial}{\partial\xi_1}\left(\varphi\frac{\partial\varphi^*}{\partial\xi_1}\right)+\tilde{\delta}^{(6)}|\varphi|^2\varphi\frac{\partial\varphi^*}{\partial\xi_1}+\mathrm{cc}\right]$$
$$+\frac{1}{2}\frac{\mathrm{d}^2\tilde{\alpha}^{(2)}}{\mathrm{d}\kappa^2}\frac{\partial\varphi}{\partial\xi_1}\frac{\partial\varphi^*}{\partial\xi_1}+\tilde{\varepsilon}^{(6)}|\varphi|^6+\tilde{\zeta}^{(6)}|\varphi|^4$$

where

$$\begin{aligned}
&\tilde{\gamma}_i^{(6)}=(V_gA_0-B_0)_{ij}\beta_j^{(4)}+g_i\\
&\tilde{\delta}_i^{(6)}=2(V_gA_0-B_0)_{ij}\gamma_j^{(4)}+h_i\\
&\tilde{\varepsilon}_i^{(6)}=\mathrm{i}q_2[(A_0)_{ij}\alpha_j^{(2)}+(\nabla A_0)_{ijk}D_{rj}D_{rk}^*]+j_i+\mathrm{cc}\\
&\tilde{\zeta}_i^{(6)}=\mathrm{i}q_1[(A_0)_{ij}\alpha_j^{(2)}+(\nabla A_0)_{ijk}D_{rj}D_{rk}^*]+\mathrm{cc}\\
&h_i-h_i^*=\mathrm{i}\,\mathrm{d}\tilde{\gamma}_i^{(4)}/\mathrm{d}\kappa+\{q_4[(A_0)_{ij}\alpha_j^{(2)}+(\nabla A_0)_{ijk}D_{rj}D_{rk}^*]+k_i-\mathrm{cc}\}
\end{aligned} \tag{3.26}$$

and g_i, j_i and k_i are given in appendix 3. We have not given an expression for h_i itself; it is sufficient here to note that it involves the $\mu_i^{(4)}$ and $\mu_i^{(4)*}$ but not the $\lambda_i^{(4)}$. From (3.1) and (3.9) we already have $\mathrm{d}^2\tilde{\alpha}^{(2)}/\mathrm{d}\kappa^2=0$ and $\tilde{\beta}_i^{(4)}=0$ for $i=1,\ldots,n-m$, so the condition (2.16) applied to $\tilde{U}_0^{(6)}$ requires that

$$\tilde{\gamma}_i^{(6)}=0 \qquad i=1,\ldots,n-m \tag{3.27}$$
$$\tilde{\delta}_i^{(6)}=0 \qquad i=1,\ldots,n-m \tag{3.28}$$
$$\tilde{\varepsilon}_i^{(6)}=0 \qquad i=1,\ldots,n-m \tag{3.29}$$
$$\tilde{\zeta}_i^{(6)}=0 \qquad i=1,\ldots,n-m. \tag{3.30}$$

On incorporating the expression (3.18) for $\beta_i^{(4)}$, we may rearrange (3.27) to give

$$\mu_i^{(4)}=\sum_{j=1}^{n-m}(F)_{ij}^{-1}\left(g_j-\sum_{k=n-m+1}^{n}g_{jk}\tilde{\beta}_k^{(4)}\right) \qquad i=1,\ldots,n-m.$$

Now q_3, q_4 and h_i are completely determined. On incorporating the expression (3.19) for $\gamma_i^{(4)}$, we may rearrange (3.28) to give

$$\lambda_i^{(4)} = \sum_{j=1}^{n-m} (F)_{ij}^{-1}\left(\tfrac{1}{2}h_j - \sum_{k=n-m+1}^{n} g_{jk}\tilde{\gamma}_k^{(4)}\right) \qquad i=1,\ldots,n-m. \tag{3.31}$$

Now q_2 and $\tilde{\varepsilon}^{(6)}$ are completely determined. For the $\lambda_i^{(4)}$, $i=1,\ldots,n-m$, to be real as required, h_i must be real in (3.31) and hence we require

$$h_i - h_i^* = 0 \qquad i=1,\ldots,n-m \tag{3.32}$$

where $h_i - h_i^*$ is given by (3.26). Note that the term i $\mathrm{d}\tilde{\gamma}_i^{(4)}/\mathrm{d}\kappa$ in (3.26) is already zero for $i=1,\ldots,n-m$, by virtue of (3.10). For a purely dispersive system it is expected that q_1, q_2, q_3 and q_4 are real, as in Kakutani and Michihiro (1983) and Parkes (1987a, b). In this case (3.30) reduces to

$$(\nabla A_0)_{ijk}(D_{rj}D_{rk}^* - D_{rj}^*D_{rk}) = 0 \tag{3.33}$$

and then (3.29) and (3.32) become, respectively,

$$j_i + j_i^* = 0 \tag{3.34}$$

$$k_i - k_i^* = 0. \tag{3.35}$$

We conclude that for a type-2 system (i.e. for $m<n$), the MNS equation is the governing equation for φ near marginal instability provided the conditions (3.1), (3.10) and (3.33)-(3.35) are satisfied. All these conditions arise essentially from the condition (2.16). In turn this condition is relevant only when $m<n$. It follows that, for a type-1 system (i.e. for $m=n$), the derivation of the MNS equation proceeds straightforwardly without the introduction of extra constraints on A, B and C.

4. Concluding remarks

Inoue and Matsumoto (1974) investigated the slow modulations of a weakly non-linear quasimonochromatic wave solution to the system (1.1) away from marginal instability. For a type-1 system the modulations are governed by the NS equation. Type-2 systems divide into types 2A and 2B. For a type-2A system the condition (3.1) holds and the NS equation is obtained. For a type-2B system the condition (3.1) does not hold and a single governing equation for the modulations cannot be obtained.

In this paper we have considered the behaviour of modulations near marginal instability. For a type-1 system the modulations are governed by the MNS equation and not the NS equation. The same is true for a type-2A system provided the extra conditions (3.10) and (3.33)-(3.35) hold. (The system describing ion acoustic wave propagation in a plasma, and considered by Parkes (1987b), falls into this category.) If any of these conditions or (3.1) do not hold, then a single governing equation for the modulations cannot be obtained.

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Appendix 1

The notation used here is explained by example:

$$(\nabla\nabla A_0)_{ijkl} \equiv \partial^2 a_{ij}/\partial u_k \partial u_l$$

evaluated at $U = U^{(0)}$, and $[(\nabla\nabla A_0)U^{(1)}U^{(2)}]_{ij} \equiv (\nabla\nabla A_0)_{ijkl}u_k^{(1)}u_l^{(2)}$ (with summation convention). Similarly

$$(\nabla\nabla C_0)_{ijk} \equiv \partial^2 c_i/\partial u_j\,\partial u_k$$

evaluated at $U = U^{(0)}$, and $[(\nabla\nabla C_0)U^{(1)}U^{(2)}]_i \equiv (\nabla\nabla C_0)_{ijk}u_j^{(1)}u_k^{(2)}$ (with summation convention). Using this notation the Taylor series for $A(U)$ about $U = U^{(0)}$ may be written

$$A = \sum_{i=0}^{5} \varepsilon^{i/2} A_i + \mathrm{O}(\varepsilon^3)$$

where

$$A_0 = A(U^{(0)})$$

$$A_1 = (\nabla A_0)U^{(1)}$$

$$A_2 = (\nabla A_0)U^{(2)} + \tfrac{1}{2}(\nabla\nabla A_0)U^{(1)}U^{(1)}$$

$$A_3 = (\nabla A_0)U^{(3)} + (\nabla\nabla A_0)U^{(1)}U^{(2)} + \tfrac{1}{6}(\nabla\nabla\nabla A_0)U^{(1)}U^{(1)}U^{(1)}$$

$$A_4 = (\nabla A_0)U^{(4)} + (\nabla\nabla A_0)(U^{(1)}U^{(3)} + \tfrac{1}{2}U^{(2)}U^{(2)}) + \tfrac{1}{2}(\nabla\nabla\nabla A_0)U^{(1)}U^{(1)}U^{(2)} + \tfrac{1}{24}(\nabla\nabla\nabla\nabla A_0)U^{(1)}U^{(1)}U^{(1)}U^{(1)}$$

$$A_5 = (\nabla A_0)U^{(5)} + (\nabla\nabla A_0)(U^{(1)}U^{(4)} + U^{(2)}U^{(3)}) + \tfrac{1}{2}(\nabla\nabla\nabla A_0)(U^{(1)}U^{(1)}U^{(3)} + U^{(1)}U^{(2)}U^{(2)}) + \tfrac{1}{6}(\nabla\nabla\nabla\nabla A_0)U^{(1)}U^{(1)}U^{(1)}U^{(2)} + \tfrac{1}{120}(\nabla\nabla\nabla\nabla\nabla A_0)U^{(1)}U^{(1)}U^{(1)}U^{(1)}U^{(1)}.$$

$B(U)$ and $C(U)$ are expanded in a similar way. Substitution of the Taylor series expansion for $C(U)$ into (2.3) gives (2.7), where

$$M_2 = -\tfrac{1}{2}(\nabla\nabla C_0)U^{(1)}U^{(1)}$$

$$M_3 = -(\nabla\nabla C_0)U^{(1)}U^{(2)} - \tfrac{1}{6}(\nabla\nabla\nabla C_0)U^{(1)}U^{(1)}U^{(1)}$$

$$M_4 = -(\nabla\nabla C_0)(U^{(1)}U^{(3)} + \tfrac{1}{2}U^{(2)}U^{(2)}) - \tfrac{1}{2}(\nabla\nabla\nabla C_0)U^{(1)}U^{(1)}U^{(2)} - \tfrac{1}{24}(\nabla\nabla\nabla\nabla C_0)U^{(1)}U^{(1)}U^{(1)}U^{(1)}$$

$$M_5 = -(\nabla\nabla C_0)(U^{(1)}U^{(4)} + U^{(2)}U^{(3)}) - \tfrac{1}{2}(\nabla\nabla\nabla C_0)(U^{(1)}U^{(1)}U^{(3)} + U^{(1)}U^{(2)}U^{(2)}) - \tfrac{1}{6}(\nabla\nabla\nabla\nabla C_0)U^{(1)}U^{(1)}U^{(1)}U^{(2)} - \tfrac{1}{120}(\nabla\nabla\nabla\nabla\nabla C_0)U^{(1)}U^{(1)}U^{(1)}U^{(1)}U^{(1)}$$

$$M_6 = -(\nabla\nabla C_0)(U^{(1)}U^{(5)} + U^{(2)}U^{(4)} + \tfrac{1}{2}U^{(3)}U^{(3)}) - (\nabla\nabla\nabla C_0)(\tfrac{1}{2}U^{(1)}U^{(1)}U^{(4)} + U^{(1)}U^{(2)}U^{(3)} + \tfrac{1}{6}U^{(2)}U^{(2)}U^{(2)}) - (\nabla\nabla\nabla\nabla C_0)(\tfrac{1}{6}U^{(1)}U^{(1)}U^{(1)}U^{(3)} + \tfrac{1}{4}U^{(1)}U^{(1)}U^{(2)}U^{(2)}) - \tfrac{1}{24}(\nabla\nabla\nabla\nabla\nabla C_0)U^{(1)}U^{(1)}U^{(1)}U^{(1)}U^{(2)} - \tfrac{1}{720}(\nabla\nabla\nabla\nabla\nabla\nabla C_0)U^{(1)}U^{(1)}U^{(1)}U^{(1)}U^{(1)}U^{(1)}.$$

Appendix 2. Expressions for V_g and $dV_g/d\kappa$

From (2.11) it follows that $D_{is}\,d(d_{ij}D_{rj})/d\kappa = 0$. On using (2.12) we obtain $D_{rj}D_{is}\,d(d_{ij})/d\kappa = 0$ and hence that

$$D_{ij}\,d(d_{ij})/d\kappa = 0 \tag{A2.1}$$

where we have used Jacobi's formula

$$D_{rj}D_{is} = D_{ij}D_{rs} \tag{A2.2}$$

and we have assumed that $D_{rs} \neq 0$. A rearrangement of (A2.1) gives

$$V_g = (B_0)_{ij}D_{ij}/\nu \tag{A2.3}$$

where $\nu = (A_0)_{ij}D_{ij}$.

By differentiating (A2.1) with respect to κ and rearranging we obtain

$$i\frac{dV_g}{d\kappa} = \frac{d}{d\kappa}(d_{ij})\frac{d}{d\kappa}(D_{ij})/\nu.$$

Also, from (3.23), we have

$$ip = \frac{d}{d\kappa}(d_{ij})\frac{d}{d\kappa}(D_{rj})D_{is}/(\nu D_{rs}).$$

Hence, in order to show that $p = \frac{1}{2}dV_g/d\kappa$, we need to prove that

$$\frac{d}{d\kappa}(d_{ij})\frac{d}{d\kappa}(D_{rj})D_{is} = \frac{1}{2}\frac{d}{d\kappa}(d_{ij})\frac{d}{d\kappa}(D_{ij})D_{rs}. \tag{A2.4}$$

Inoue and Matsumoto (1974, appendices D and E) proved (A2.4) using a complicated procedure involving determinants $D_{ij,kl}$ that are related to the matrix obtained from (d_{ij}) be deleting the ith and kth rows and jth and lth columns. Here we present a simpler and shorter proof.

Denoting the left-hand side of (A2.4) by H, and using the derivatives of (2.11) and (2.12) with respect to κ, we have

$$H = -d_{ij}\frac{d}{d\kappa}(D_{rj})\frac{d}{d\kappa}(D_{is}) = \frac{d}{d\kappa}(d_{ij})\frac{d}{d\kappa}(D_{is})D_{rj}.$$

Hence we may write

$$H = \frac{1}{2}\frac{d}{d\kappa}(d_{ij})\frac{d}{d\kappa}(D_{rj}D_{is}).$$

The desired result (A2.4) follows on using (A2.2) and then (A2.1).

Appendix 3

The vector coefficients that are not given explicitly in § 3 are stated here. First we introduce some convenient notation, namely

$$q_{ijk}=\mathrm{i}(\omega\nabla A_0-\kappa\nabla B_0)_{ijk} \qquad r_{ijk}=q_{ijk}-(\nabla\nabla C_0)_{ijk}$$
$$s_{ijk}=q_{ijk}-\tfrac{1}{2}(\nabla\nabla C_0)_{ijk} \qquad t_{ijk}=q_{ijk}-\tfrac{1}{3}(\nabla\nabla C_0)_{ijk}$$
$$u_{ijk}=q_{ijk}-\tfrac{1}{4}(\nabla\nabla C_0)_{ijk} \qquad v_{ijk}=q_{ijk}-\tfrac{1}{6}(\nabla\nabla C_0)_{ijk}$$
$$w_{ijk}=(-V_g\nabla A_0+\nabla B_0)_{ijk}.$$

The corresponding higher-order tensors are defined in the obvious way, for example

$$q_{ijkl}=\mathrm{i}(\omega\nabla\nabla A_0-\kappa\nabla\nabla B_0)_{ijkl} \qquad r_{ijkl}=q_{ijkl}-(\nabla\nabla\nabla C_0)_{ijkl}.$$

Now we have

$$\tilde{\alpha}_i^{(2)}=s_{ijk}D_{rj}D_{rk}^*+\mathrm{cc}\quad \tilde{\beta}_i^{(2)}=s_{ijk}D_{rj}D_{rk}$$
$$\tilde{\gamma}_i^{(3)}=r_{ijk}[D_{rj}\alpha_k^{(2)}+2D_{rk}^*\beta_j^{(2)}-D_{rj}^*\beta_k^{(2)}]+\tfrac{1}{2}r_{ijkl}[2D_{rj}D_{rk}^*-D_{rj}^*D_{rk}]D_{rl}$$
$$\tilde{\delta}_i^{(3)}=t_{ijk}[D_{rj}\beta_k^{(2)}+2D_{rk}\beta_j^{(2)}]+\tfrac{1}{2}t_{ijkl}D_{rj}D_{rk}D_{rl}$$
$$\tilde{\alpha}_i^{(4)}=-[(A_0)_{ij}\alpha_j^{(2)}+(\nabla A_0)_{ijk}D_{rj}^*D_{rk}]-q_{ijk}D_{rk}\alpha_j^{(3)*}+r_{ijk}D_{rj}\alpha_k^{(3)*}$$
$$\begin{aligned}\tilde{\gamma}_i^{(4)}=\;&s_{ijk}[D_{rk}^*\gamma_j^{(3)}+D_{rj}\gamma_k^{(3)*}]+u_{ijk}[\alpha_j^{(2)}\alpha_k^{(2)}+2\beta_j^{(2)}\beta_k^{(2)*}]+s_{ijkl}D_{rj}D_{rk}^*\alpha_l^{(2)}\\&+u_{ijkl}[D_{rj}D_{rk}\beta_l^{(2)*}+D_{rk}^*D_{rl}^*\beta_j^{(2)}]+\tfrac{1}{2}u_{ijklm}D_{rj}D_{rk}D_{rl}^*D_{rm}^*+\mathrm{cc}\end{aligned}$$
$$f_i=-\mathrm{i}q_{ijk}D_{rk}\,\mathrm{d}(D_{rj}^*)/\mathrm{d}\kappa+\mathrm{i}r_{ijk}D_{rj}\,\mathrm{d}(D_{rk}^*)/\mathrm{d}\kappa-w_{ijk}D_{rj}^*D_{rk}$$
$$\begin{aligned}\tilde{\varepsilon}_i^{(4)}=\;&s_{ijk}[2\alpha_k^{(2)}\beta_j^{(2)}+D_{rk}\gamma_j^{(3)}+D_{rj}\gamma_k^{(3)}+3D_{rk}^*\delta_j^{(3)}-D_{rj}^*\delta_k^{(3)}]\\&+s_{ijkl}[D_{rj}D_{rk}\alpha_l^{(2)}+2D_{rk}^*D_{rl}\beta_j^{(2)}+(D_{rj}D_{rk}^*-D_{rk}D_{rj}^*)\beta_l^{(2)}]\\&+\tfrac{1}{6}s_{ijklm}[3D_{rj}D_{rk}^*-D_{rj}^*D_{rk}]D_{rl}D_{rm}\end{aligned}$$
$$\begin{aligned}\tilde{\delta}_i^{(5)}=\;&(V_gA_0-B_0)_{ij}\gamma_j^{(3)}+r_{ijk}\{D_{rj}\beta_k^{(4)}+\mathrm{i}[2\beta_j^{(2)}\,\mathrm{d}(D_{rk}^*)/\mathrm{d}\kappa-\beta_k^{(2)}\,\mathrm{d}(D_{rj}^*)/\mathrm{d}\kappa]\}\\&-w_{ijk}[D_{rk}\alpha_j^{(2)}+D_{rj}^*\beta_k^{(2)}]+\tfrac{1}{2}\mathrm{i}r_{ijkl}[2D_{rj}D_{rk}\,\mathrm{d}(D_{rl}^*)/\mathrm{d}\kappa\\&-D_{rk}D_{rl}\,\mathrm{d}(D_{rj}^*)/\mathrm{d}\kappa]-\tfrac{1}{2}w_{ijkl}D_{rj}^*D_{rk}D_{rl}\end{aligned}$$
$$\tilde{\varepsilon}_i^{(5)}=(V_gA_0-B_0)_{ij}\gamma_j^{(3)}-\mathrm{i}r_{ijk}D_{rj}\sigma_k-\mathrm{i}\,\mathrm{d}\tilde{\gamma}_i^{(3)}/\mathrm{d}\kappa$$
$$\begin{aligned}\tilde{\zeta}_i^{(5)}=\;&r_{ijk}[\gamma_j^{(3)}\alpha_k^{(2)}+3\delta_j^{(3)}\beta_k^{(2)*}-2\delta_k^{(3)}\beta_j^{(2)*}+2\beta_j^{(2)}\gamma_k^{(3)*}-\beta_k^{(2)}\gamma_j^{(3)*}\\&+D_{rj}\gamma_k^{(4)}+2D_{rk}^*\varepsilon_j^{(4)}-D_{rj}^*\varepsilon_k^{(4)}]\\&+\tfrac{1}{2}r_{ijkl}[D_{rj}(\alpha_k^{(2)}\alpha_l^{(2)}+2\beta_k^{(2)}\beta_l^{(2)*})+4D_{rk}(\beta_j^{(2)}\beta_l^{(2)*}-\beta_l^{(2)}\beta_j^{(2)*})\\&+4D_{rk}^*\alpha_l^{(2)}\beta_j^{(2)}-2D_{rj}^*\alpha_k^{(2)}\beta_l^{(2)}+2D_{rk}D_{rl}^*\gamma_j^{(3)}+2(D_{rj}D_{rk}^*-D_{rj}^*D_{rk})\gamma_l^{(3)}\\&+D_{rk}(2D_{rj}\gamma_l^{(3)*}-D_{rl}\gamma_j^{(3)*})+D_{rk}^*(3D_{rl}^*\delta_j^{(3)}-2D_{rj}^*\delta_l^{(3)})]\\&+\tfrac{1}{6}r_{ijklm}[3(2D_{rj}D_{rk}^*-D_{rj}^*D_{rk})D_{rl}\alpha_m^{(2)}+6D_{rk}^*D_{rl}^*D_{rm}\beta_j^{(2)}\\&-3(2D_{rj}^*D_{rk}-D_{rj}D_{rk}^*)D_{rl}^*\beta_m^{(2)}+(3D_{rj}\beta_m^{(2)*}-2D_{rm}\beta_j^{(2)*})D_{rl}D_{rk}]\\&+\tfrac{1}{24}r_{ijklmn}[6D_{rj}D_{rk}^*-4D_{rj}^*D_{rk}]D_{rl}D_{rm}D_{rn}^*\end{aligned}$$
$$\begin{aligned}g_i=\;&\mathrm{i}p[(A_0)_{ij}\alpha_j^{(2)}+(\nabla A_0)_{ijk}D_{rj}^*D_{rk}]+\tfrac{1}{2}q_{ijk}D_{rk}\,\mathrm{d}^2(D_{rj}^*)/\mathrm{d}\kappa^2\\&-\tfrac{1}{2}r_{ijk}D_{rj}\,\mathrm{d}^2(D_{rk}^*)/\mathrm{d}\kappa^2-\mathrm{i}w_{ijk}D_{rk}\,\mathrm{d}(D_{rj}^*)/\mathrm{d}\kappa\end{aligned}$$

$$
\begin{aligned}
j_i = {} & s_{ijk}[D^*_{rk}(\zeta^{(5)}_j - \mathrm{i}q_2\alpha^{(3)}_j) + D_{rj}(\zeta^{(5)}_k - \mathrm{i}q_2\alpha^{(3)}_k)^* + \alpha^{(2)}_j\gamma^{(4)}_k + \gamma^{(3)}_j\gamma^{(3)*}_k] \\
& + 2u_{ijk}[\varepsilon^{(4)}_j\beta^{(2)*}_k + \beta^{(2)}_j\varepsilon^{(4)*}_k] + 3v_{ijk}\delta^{(3)}_j\delta^{(3)*}_k \\
& + s_{ijkl}[D_{rj}D^*_{rk}\gamma^{(4)}_l + (D^*_{rk}\gamma^{(3)}_j + D_{rj}\gamma^{(3)*}_k)\alpha^{(2)}_l \\
& + D^*_{rk}(3\delta^{(3)}_j\beta^{(2)*}_l - 2\delta^{(3)}_l\beta^{(2)*}_j) + D_{rj}\beta^{(2)}_k\delta^{(3)*}_l] \\
& + u_{ijkl}[D^*_{rk}D^*_{rl}\varepsilon^{(4)}_j + D_{rj}D_{rk}\varepsilon^{(4)*}_l - 2\alpha^{(2)}_l\beta^{(2)}_j\beta^{(2)*}_k \\
& + D^*_{rk}(\beta^{(2)}_l\gamma^{(3)*}_j + 2\beta^{(2)}_j\gamma^{(3)*}_l) \\
& + D_{rj}\gamma^{(3)}_k\beta^{(2)*}_l + \tfrac{1}{3}\alpha^{(2)}_j\alpha^{(2)}_k\alpha^{(2)}_l] \\
& + \tfrac{1}{2}t_{ijklm}[D_{rk}D^*_{rl}D^*_{rm}\gamma^{(3)}_j + (3D_{rj}D^*_{rk} - 2D^*_{rj}D_{rk})D_{rl}\gamma^{(3)*}_m] \\
& + u_{ijklm}[2D_{rj}D^*_{rk}\beta^{(2)}_l\beta^{(2)*}_m + D_{rl}D^*_{rm}(\beta^{(2)}_j\beta^{(2)*}_k - \beta^{(2)}_k\beta^{(2)*}_j) \\
& + D_{rj}D^*_{rk}\alpha^{(2)}_l\alpha^{(2)}_m + (D^*_{rk}D^*_{rl}\beta^{(2)}_j + D_{rj}D_{rk}\beta^{(2)*}_l)\alpha^{(2)}_m] \\
& + \tfrac{1}{2}v_{ijklm}[D^*_{rk}D^*_{rl}D^*_{rm}\delta^{(3)}_j + D_{rj}D_{rk}D_{rl}\delta^{(3)*}_m] \\
& + \tfrac{1}{6}u_{ijklmn}[3D_{rj}D_{rk}D^*_{rl}D^*_{rm}\alpha^{(2)}_n + 2D^*_{rk}D^*_{rl}D^*_{rm}D_{rn}\beta^{(2)}_j \\
& + (3D_{rj}D^*_{rk} - D^*_{rj}D_{rk})D_{rl}D_{rm}\beta^{(2)*}_n] \\
& + \tfrac{1}{12}v_{ijklmnp}D_{rj}D_{rk}D_{rl}D^*_{rm}D^*_{rn}D^*_{rp} \\
k_i = {} & s_{ijk}[D_{rj}(\varepsilon^{(5)}_k + \mathrm{i}\,\mathrm{d}\gamma^{(3)}_k/\mathrm{d}\kappa + q_4\alpha^{(3)}_k)^* - D^*_{rk}(\varepsilon^{(5)}_j + \mathrm{i}\,\mathrm{d}\gamma^{(3)}_j/\mathrm{d}\kappa + q_4\alpha^{(3)}_j) \\
& + \mathrm{i}\alpha^{(2)}_j\sigma_k] + \mathrm{i}s_{ijkl}D_{rj}D^*_{rk}\sigma_l.
\end{aligned}
$$

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